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Some unsteady motions of a diffusion flame sheet

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A study is made of the response of a diffusion flame sheet to perturbations of reactant concentration which are introduced, either as changes in the free streams, or as appropriate initial distributions throughout the field. Species' and energy conservation requirements are approximated by linearized boundary-layer equations; general solutions are derived for species, enthalpy and temperature distributions, as well as for the flame sheet shape, and a number of specific problems are solved.

1. Introduction

It is inevitable that any theoretical account of a phenomenon with as many interconnected physical features as a diffusion flame must rely on a number of simplifying assumptions especially when time-dependent effects are involved. The precise nature of such assumptions will depend, to some extent, on the geometrical configuration to be studied. In general three configurations are important; the counter-flow (or 'stagnation point') flame, the cylindrical arrangement, with a tube-ejecting fuel into oxidizer carried in a second coaxial pipe, for example, and the (substantially) plane flame arising above a properly designed flat flame burner. It is with the last of these three geometries that we shall be most concerned here, although the distinctions between it and the second form diminish as the ratio of the distance from the fuel pipe outlet to its diameter decreases.

The planar, two-dimensional layout to be studied is sketched in figure 1. Both oxidizer and fuel streams have the same constant speed U and are at the same pressure. They are kept apart by a thin partition which terminates at $x' = 0$. Beyond this point, mixing and combustion take place. Both free streams may be diluted with some inert species. The partition is assumed to produce no disturbance to the flow field, so that linearization of convective terms in the conservation equations is in order. (Heat release in the flame has comparatively small effect on flow velocities in such an arrangement; see, for example, Clarke, 1967*a*). A consistent assumption is that the work done by the pressure and viscous stresses is small and can be omitted from the energy equation. If the coefficients of mass and heat diffusion are now assumed to be constant and equal, the species' conservation and energy equations can be dealt with separately and furthermore (chemical 'source' terms apart) are linear. It follows that any perturbations to

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the steady state will arise, not from hydrodynamic sources, but from compositional deviations from the steady regime. Constancy and equality of diffusion coefficients is the most radical of the assumptions listed; for the present it seems justifiable on the grounds of considerable analytical simplicity, plus the fact that no gross distortion of the basic physical mechanisms is involved.

In the present paper we adopt the Burke–Schumann sheet model for the diffusion flame. This can be viewed as the limiting case of an infinitely fast irreversible 'oxidant-plus-fuel-equals-product' type of reaction. Since the recent work of Fendell (1965) this limit in steady flows has become firmly established as the 'first outer solution' in an asymptotic representation of the flame field for large burning rates. This is demonstrated, for example, in the articles by Fendell & Chung (1965) and Chung & Blankenship (1966) which deal with counter-flow flames, and by Clarke (1967*b*) for flames for the type considered here. It has been shown by Clarke (1968) that first outer solutions of the Burke–Schumann type are valid for a steady hydrogen–oxygen flame supported by a moderately realistic set of chain reactions, and that a subsequent inner solution leads to quite good predictions of flame structure. This may be interpreted as widening the applicability of simple flame sheet solutions away from the rather naïve type of reaction kinetics mentioned above. It is, of course, necessary to establish the conditions which apply across a diffusion flame sheet in *unsteady* motion, since (to our knowledge) they have not so far been explicitly stated. For this purpose the simplified kinetic scheme is adequate; the extension of arguments (like those given in §3 of the paper by Clarke (1967*a*), for example) to include time-dependent terms is immediate and requires no special treatment here. It is sufficient to note that the flame sheet is specified by precisely the same conditions (namely, vanishing reactant concentrations, and normal concentration-gradient jumps in stoichiometric ratio) as those found in a steady situation.

With the assumptions made so far, the situation on either side of the flame sheet is described by equations of the form

$$\kappa\nabla'^2\psi - U\psi_{x'} - \psi_{t'} = 0, \tag{1}$$

where κ is the diffusivity, ∇'^2 is the Laplacian operator in x', y' co-ordinates and ψ is either a mass fraction c_α, for the species α, or the specific enthalpy of the mixture, h. The apparent simplicity of (1) is deceptive, particularly when viewed in the context of the present problem. The relevant fundamental solution (or Green's function) is a comparatively intractable quantity so that solutions in the present case, while not impossible, are very unwieldy. The simple, separable, solutions used in the steady flow case (Clarke 1967*b*) are also not available for the particular problem to be considered here. Accordingly, it is most helpful to introduce a further approximation, of the boundary-layer type, at this stage. This consists simply of retaining the term $\psi_{y'y'}$ in $\nabla'^2\psi$, and therefore of neglecting the x'-wise diffusion effects summarized by the term $\psi_{x'x'}$. Such an approximation is consistent with the idea that the flame is flat and lies substantially parallel to the x'-axis. This has certainly been found to be the case for steady flames in circumstances like those under investigation here (Clarke 1967*b*), with the exception of a region in the vicinity of the origin. There the parabolic shape of the flame has been shown to be of the form

$$y' = a(x' + \tfrac{1}{4}a^2)^{\frac{1}{2}}, \tag{2}$$

where a is a constant depending on the parameters of the problem; in particular it is a numerical constant multiplied by $(\kappa/U)^{\frac{1}{2}}$. With the boundary-layer approximation introduced above the steady-flame shape would be given by

$$y' = ax'^{\frac{1}{2}} \tag{3}$$

(this will be demonstrated below), so that our approximate form of (1), namely

$$\kappa\psi_{y'y'} - U\psi_{x'} - \psi_{t'} = 0 \tag{4}$$

gives solutions comparable in steady flows with those of equation (1) for

$$x' \gg -\tfrac{1}{4}a^2.$$

We propose, at this stage, to tolerate any singular behaviour in the solutions as $x' \downarrow 0$.

Equations similar to (4) (often called linearized boundary-layer equations) have been used previously in heuristic discussions of velocity boundary-layer behaviour. Lagerstrom, Cole & Trilling (1949) mention the steady-flow case, and Stewartson (1951) has used it to examine unsteady velocity boundary-layer conditions for flow over a semi-infinite flat plate started impulsively from rest. From the latter analysis we anticipate some discontinuous behaviour in the solutions on crossing the line $x' = Ut'$. Physically this arises from the absence of the diffusive term $\psi_{x'x'}$, which would act to smooth out any discontinuous behaviour sanctioned by the wave-like final terms in (4). (These represent a wave propagating from left to right with constant speed U.) Once again, for the present, we propose to tolerate this physical deficiency. It is worth remarking that, while the practical utility of (4) is marred for the velocity boundary layer by the need to satisfy the no-slip condition at the solid surface, no such objection can be raised to the use of linear convective terms in the present case. The flame exists in the homogeneous gas phase far removed (except near $x' = 0$) from such gross velocity perturbations and, in the circumstances, we anticipate that local variations in convection speed will be of secondary importance.

It is clearly convenient to work in terms of dimensionless independent variables, so that we define

$$x = x'U/\kappa, \quad y = y'U/\kappa, \quad t = t'U^2/\kappa. \tag{5}$$

Equation (4) is therefore equivalent to

$$\psi_{yy} - \psi_x - \psi_t = 0, \tag{6}$$

where ψ is either c_O (oxidant mass fraction), c_F (fuel mass fraction), c_i (mass fraction of the inert diluent), or h (the specific enthalpy).

The flame sheet is located at

$$y = y_f(x, t), \tag{7}$$

and it is part of our task to calculate the function y_f for given initial and boundary conditions. The behaviour of the mass fractions at the flame sheet is conditioned by the requirements that

$$c_O(x, y_f+, t) = 0 = c_F(x, y_f-, t), \tag{8}$$

$$\frac{\partial}{\partial y} c_O(x, y_f+, t) = -\mathscr{M} \frac{\partial}{\partial y} c_F(x, y_f-, t), \tag{9}$$

where $\mathscr{M}$ is the stoichiometric oxidant-to-fuel mass-fraction ratio. c_i must be continuous and have continuous y-derivatives across $y = y_f$.

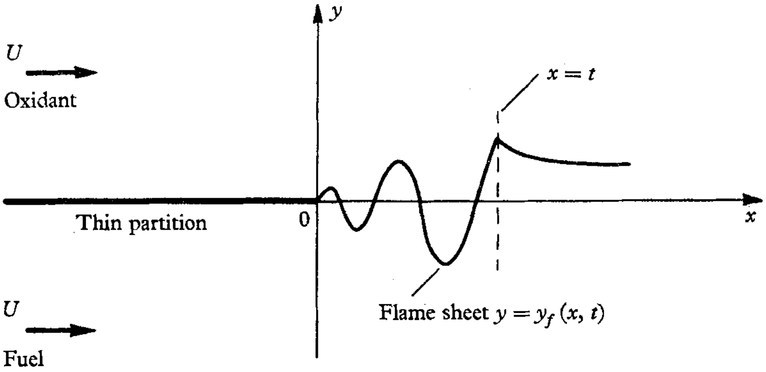

Figure 1. Sketch of the configuration. x, y and t are the dimensionless co-ordinates defined in (5). The flame sheet is shown in a position corresponding to oscillatory free stream concentrations (between $x = 0$ and $x = t$) and non-steady initial conditions ($x > t$).

A few words of explanation of (9) are perhaps in order. The basic requirement (e.g. Clarke 1967*a*) is for the derivatives of c_O and c_F *normal* to the flame sheet to obey a relation of the form (9). Writing y_{fx} for the (partial) x-derivative of y_f, the normal derivative operator is equal to

$$(1+y_{fx}^2)^{-\frac{1}{2}} \left\{\frac{\partial}{\partial y} - y_{fx}\frac{\partial}{\partial x}\right\},$$

and the tangential derivative operator is

$$(1+y_{fx}^2)^{-\frac{1}{2}} \left\{\frac{\partial}{\partial x} + y_{fx}\frac{\partial}{\partial y}\right\}.$$

But the mass fractions c_O and c_F are constant (zero) along the flame so that, from the latter form, one has

$$\frac{\partial}{\partial x} c_\alpha = -y_{fx}\frac{\partial}{\partial y} c_\alpha; \quad \alpha = O, F.$$

The normal derivative of c_α is therefore equal to

$$(1+y_{fx}^2)^{+\frac{1}{2}} \frac{\partial}{\partial y} c_\alpha$$

and (9) follows.

With the assumptions of no pressure-work and equality of mass and heat diffusion coefficients, the equations satisfied by h and c_i are homogeneous and both variables are continuous with continuous first and second y-derivatives everywhere. The same can be said of c_O and c_F in any region where these quantities satisfy equations like (6). Referring to figure 1, the remarks can be taken to apply to c_O in the half-plane $y > y_f$ and to c_F in $y < y_f$. The analysis is greatly facilitated if the conditions (8) and (9) can be applied, not on $y = y_f$, but on $y = 0$. It is therefore important to note that, since c_O and c_F will have to satisfy (6) in $y > 0$ or $y < 0$, respectively, with the consequence that their y-derivatives are well-behaved, it is possible to use Taylor series for c_O and c_F, as follows:

$$c_O(x, y_f+, t) = c_O(x, 0+, t) + \frac{\partial}{\partial y} c_O(x, 0+, t) y_f + \ldots,$$

$$c_F(x, y_f-, t) = c_F(x, 0-, t) + \frac{\partial}{\partial y} c_F(x, 0-, t) y_f + \ldots.$$

From (8) it follows that, to first order in y_f,

$$c_O(x, 0+, t) = -y_f F_O(x, 0+, t), \tag{10}$$

$$c_F(x, 0-, t) = -y_f F_F(x, 0-, t), \tag{11}$$

where

$$F_O \equiv \frac{\partial}{\partial y} c_O(x, 0+, t), \tag{12}$$

$$F_F \equiv \frac{\partial}{\partial y} c_F(x, 0-, t). \tag{13}$$

The stoichiometry condition (9) can similarly be replaced by

$$F_O = -\mathscr{M} F_F. \tag{14}$$

The solutions giving mass fraction and enthalpy distributions, and the location $y_f(x,t)$ of the flame sheet, can now be obtained from a consideration of the boundary and initial value problem for (6).

2. Some general solutions

Green's function for (6) is defined as a solution of

$$G_{yy} - G_x - G_t = -4\pi\delta(x-x_0)\,\delta(y-y_0)\,\delta(t-t_0), \tag{15}$$

which satisfies appropriate homogeneous conditions on certain of the boundaries (just which we shall see below) in the domain within which a solution of (6) is sought. x_0, y_0, t_0 are the source co-ordinates and $\delta(z-z_0)$ is the Dirac function of $z-z_0$ ($=0, z \neq z_0$; $=\infty, z = z_0$; $z = x, y, t$). G, which we shall sometimes write as $G(x, y, t \mid x_0, y_0, t_0)$ to indicate the response at a field point x, y, t due to the presence of a source at x_0, y_0, t_0, must also satisfy a causality condition, namely

$$G(x, y, t \mid x_0, y_0, t_0) = 0 \quad (t < t_0). \tag{16}$$

Making the substitutions

$$\eta = (x-x_0) - (t-t_0), \tag{17a}$$

$$Y = y - y_0, \quad \tau = t - t_0, \tag{17b}$$

equation (15) becomes

$$G_{YY} - G_\tau = -4\pi\delta(\eta)\,\delta(Y)\,\delta(\tau). \tag{18}$$

Apart from the function $\delta(\eta)$, this is precisely the equation which defines Green's function for the one-dimensional unsteady diffusion equation. We immediately infer that, in an unbounded domain, the Green's function that we require (writing g for the unbounded domain value of G) is given by

$$g = 2\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \exp\{-Y^2/4\tau\}\delta(\eta)H(\tau), \tag{19}$$

where $H(\tau)$ is the unit function of τ ($= 0, \tau < 0$; $= 1, \tau > 0$). The form of g exhibits two essential features of our problem, namely the diffusive behaviour in y-directions combined with simple wave propagation along positive x-directions at a unit (in the dimensionless system) speed. (The source only produces an effect when $\eta = 0$; that is when $x - x_0 = \tau > 0$, where the inequality arises from $H(\tau)$ in (19). It follows that there is no upstream influence according to the present theory.) For the problems treated below, G can be found from g by simple image methods.

By the usual manipulations (e.g. Morse & Fesbach 1953) the general solution for $\psi(x,y,t)$ in the domain $x > 0, t > 0, y_a > y > y_b$ can be shown to be

$$\begin{aligned} 4\pi\psi(x,y,t) = {} & \int_0^{t+}\int_0^{x+}\left[G(x,y,t\,|\,x_0,y_0,t_0)\frac{\partial}{\partial y_0}\psi(x_0,y_0,t_0)\right. \\ & \left. -\psi(x_0,y_0,t_0)\frac{\partial}{\partial y_0}G(x,y,t\,|\,x_0,y_0,t_0)\right]_{y_0=y_b}^{y_0=y_a} dx_0\,dt_0 \\ & + \int_{y_a}^{y_b}\int_0^{t+}\psi(0,y_0,t_0)G(x,y,t\,|\,0,y_0,t_0)\,dt_0\,dy_0 \\ & + \int_{y_b}^{y_a}\int_0^{x+}\psi(x_0,y_0,0)G(x,y,t\,|\,x_0,y_0,0)\,dx_0\,dy_0. \end{aligned} \tag{20}$$

In the first double integral the notation implies that the integrand consists of the terms shown, evaluated at $y_0 = y_a$ minus the same terms evaluated at $y_0 = y_b$. There are no contributions to the solution from integration on the 'surfaces' $t_0 = t+$ and $x_0 = x+$, on account of causality.

The first two integrals in (20) combine the effects of the boundary conditions and the last integral represents the influence of initial values.

When considering solutions for c_O (in $y > 0$) we shall take $y_b = 0+$ and let $y_a \to \infty$; for c_F (in $y < 0$) we shall put $y_a = 0-$ and let $y_b \to -\infty$; for the enthalpy h and diluent mass fraction c_i we let $y_a \to \infty$ and $y_b \to -\infty$. In the two latter cases g is the correct form for G. The boundary-value integrals at y_a and y_b both vanish in the limit ($y_a \to \infty$, $y_b \to -\infty$) provided that h, c_i and their y-derivatives are bounded. It is physically desirable that these conditions should be met whence, using (19) in (20) it follows that

$$\begin{aligned} 2\pi^{\frac{1}{2}}\phi(x,y,t) = {} & \int_{-\infty}^{\infty}\phi(0,y_0,t-x)\frac{\exp\{-Y^2/4x\}}{x^{\frac{1}{2}}}\,dy_0\,H(t-x) \\ & + \int_{-\infty}^{\infty}\phi(x-t,y_0,0)\frac{\exp\{-Y^2/4t\}}{t^{\frac{1}{2}}}\,dy_0\,H(x-t), \end{aligned} \tag{21}$$

where ϕ is either h or c_i.

Since there is no upstream influence in the present description we can reasonably refer to conditions on $x = 0$ as free-stream conditions. Solution (21) shows that at positions downstream of the convection wave head at $x = t$ both h and c_i are determined by their initial distributions at a location to the left of the field point by an amount equal to the distance travelled by the wave in the time interval 0 to t (this is the second integral in (21)). To the left of $x = t$, conditions are determined by the free stream behaviour at an earlier time (first integral in (21)) which is again fixed by the convection speed. We remark that the initial and free stream data can be discontinuous; in particular neither $h(0, y_0, t_0)$ nor $c_i(0, y_0, t_0)$ will necessarily be continuous across $y_0 = 0$, for example.

3. The species' concentrations and flame sheet position

In using the general solutions (20) for c_O and c_F we must pay attention to the type of boundary value data to be used in the integrals along $y = y_a$ and y_b. It is clear that the physical requirements of boundedness for c_O, c_F and their y-derivatives are sufficient to ensure that the integrals along $y_a \to \infty$ (for c_O) and $y_b \to -\infty$ (for c_F) will vanish in the limit. Therefore it is only necessary to deal with the first integral in (20) when $y_b = 0+$ and $y_a = 0-$ for c_O and c_F respectively.

In fact we have no direct data for c_O, c_F or their y-derivatives as $|y| \to 0$, nor indeed do we know anything about combined behaviour, as might be implied from (10) and (11), since y_f is an (as yet) unknown function. We find it useful to consider two forms of solution; the first assumes that c_O and c_F are given on $y = 0$, and uses the Green function

$$G_- = 2\left(\frac{\pi}{t-t_0}\right)^{\frac{1}{2}} (\exp\{-(y-y_0)^2/4(t-t_0)\} - \exp\{-(y+y_0)^2/4(t-t_0)\})\,\delta(\eta)\,H(\tau), \quad (22)$$

while the second assumes that $\partial c_O/\partial y$ and $\partial c_F/\partial y$ are given, so that an appropriate function is

$$G_+ = 2\left(\frac{\pi}{t-t_0}\right)^{\frac{1}{2}} (\exp\{-(y-y_0)^2/4(t-t_0)\} + \exp\{-(y+y_0)^2/4(t-t_0)\})\,\delta(\eta)\,H(\tau). \quad (23)$$

Putting these forms of G into solution (20), and making use of the properties of the δ and H functions, gives two versions of the solution for, let us say, c_O:

$$\begin{aligned} 2\pi^{\frac{1}{2}} c_O(x,y,t) = &\left\{ \int_{(x-t)\,H(x-t)}^{x} c_O(x_0, 0+, x_0+t-x)\, y \,\frac{\exp\{-y^2/4(x-x_0)\}}{(x-x_0)^{\frac{3}{2}}}\, dx_0 \right. \\ \text{or} \quad &\left. -2\int_{(x-t)\,H(x-t)}^{x} F_O(x_0, 0+, x_0+t-x)\, \frac{\exp\{-y^2/4(x-x_0)\}}{(x-x_0)^{\frac{1}{2}}}\, dx_0 \right\} \\ &+ \int_0^\infty c_O(0, y_0, t-x)\,[\exp\{-(y-y_0)^2/4x\} \\ &\qquad \mp \exp\{-(y+y_0)^2/4x\}]\,\frac{dy_0}{x^{\frac{1}{2}}}\, H(t-x) \\ &+ \int_0^\infty c_O(x-t, y_0, 0)[\exp\{-(y-y_0)^2/4t\} \\ &\qquad \mp \exp\{-(y+y_0)^2/4t\}]\,\frac{dy_0}{t^{\frac{1}{2}}}\, H(x-t). \end{aligned} \quad (24)$$

The upper (minus) signs inside the square brackets in the last two integrals of (24) are to be taken with the first integral in the curly brackets; or one must take the second integral in curly brackets with the lower (plus) signs in the square brackets.

The solution for $c_F(x, y, t)$ is similar to (24) except for the following changes; replace subscript O by subscript F, multiply the integrals in curly brackets by -1 and change the limits in the last two integrals from 0 to ∞ to $-\infty$ to 0.

It is important to note that the solutions for c_O and c_F are only valid in $x > 0$, $|y| > 0$, $t > 0$.

It is now convenient to define the integrals

$$J_{\alpha\mp} = \int c_\alpha(x-t, y_0, 0)\,[\exp\{-(y-y_0)^2/4t\} \mp \exp\{-(y+y_0)^2/4t\}]\,\frac{dy_0}{t^{\frac{1}{2}}}\,H(x-t), \tag{25}$$

$$K_{\alpha\mp} = \int c_\alpha(0, y_0, t-x)\,[\exp\{-(y-y_0)^2/4x\} \mp \exp\{-(y+y_0)^2/4x\}]\,\frac{dy_0}{x^{\frac{1}{2}}}\,H(t-x), \tag{26}$$

where the limits of integration are 0 to ∞ when $\alpha = O$ and $-\infty$ to 0 when $\alpha = F$: in addition, one must take either upper or lower signs throughout, so that there are four 'J' integrals and four 'K' integrals in all. When $y = 0$ all four of $J_{\alpha-}$ and $K_{\alpha-}$ are zero; $J_{\alpha+}$ and $K_{\alpha+}$ are not zero, however, and we write

$$J^{(0)}_{\alpha+}, K^{(0)}_{\alpha+} \quad \text{to denote} \quad J_{\alpha+}, K_{\alpha+} \quad \text{in the limit} \quad |y| \to 0. \tag{27}$$

Using this notation we can now find c_O and c_F from (24) as $|y| \to 0$:

$$2\pi^{\frac{1}{2}} c_O(x, 0+, t) = -2\int_{(x-t)\,H(x-t)}^{x} F_O(x_0, 0+, x_0+t-x)\,\frac{dx_0}{(x-x_0)^{\frac{1}{2}}} + K^{(0)}_{O+} + J^{(0)}_{O+} = -2\pi^{\frac{1}{2}} y_f F_O; \tag{28}$$

$$2\pi^{\frac{1}{2}} c_F(x, 0-, t) = +2\int_{(x-\ )\,H(x-t)}^{x} F_F(x_0, 0-, x_0+t-x)\,\frac{dx_0}{(x-x_0)^{\frac{1}{2}}} + K^{(0)}_{F+} + J^{(0)}_{F+} = -2\pi^{\frac{1}{2}} y_f F_F. \tag{29}$$

The last results in (28) and (29) follow from (10) and (11).

Dividing the second version of (28) by the second version of (29) and subsequently using (14) it is readily found that

$$2\int_{(x-t)\,H(x-t)}^{x} F_O(x_0, 0+, x_0+t-x)\,\frac{dx_0}{(x-x_0)^{\frac{1}{2}}} = \tfrac{1}{2}\{K^{(0)}_{O+} + J^{(0)}_{O+} + \mathscr{M}(K^{(0)}_{F+} + J^{(0)}_{F+})\}. \tag{30}$$

Putting this result into (28) gives

$$c_O(x, 0+, t) = -y_f F_O = \frac{1}{4\pi^{\frac{1}{2}}}\{K^{(0)}_{O+} + J^{(0)}_{O+} - \mathscr{M}(K^{(0)}_{F+} + J^{(0)}_{F+})\}. \tag{31}$$

Since, in principle, the 'J' and 'K' integrals can be evaluated for any given initial and free-stream data, (31) gives the solution for $c_O(x, 0+, t)$. This is the boundary value required for a complete solution, giving c_O as a function of x, y

and t, by using the first of the two forms in (24). It follows that solutions for c_F can also be found and the problem is, formally, solved. Once $c_O(x, y, t)$ is known it is a simple matter to find F_O, whence y_f (the flame sheet position) can be deduced from (31).

It should be noted that it is the value of c_O when x is equal to x_0, y to $0+$ and t to x_0+t-x that is required in (24) (see the first integral there). Therefore one must make these substitutions in $K_{\alpha+}$ and $J_{\alpha+}$; in particular we note that the arguments $x-t$ and $t-x$ of the unit step functions are preserved by the substitutions. The inference is that the behaviour of c_O (and hence c_F) can, like that of h and c_i, be considered separately in two regions, namely $0 < x < t$ and $0 < t < x$. In the former case, only the $K_{\alpha\mp}$ integrals contribute (they describe the influence of the free-stream conditions), while in the latter case it is the $J_{\alpha\mp}$ integrals which are involved; they describe the effect of the initial conditions.

There are a number of routes to the final solution in addition to the one just mentioned. One could, for example, find F_O by solving the integral equation presented in (30), or by differentiating the general solution for c_O given in (24) (first version). Little or no advantage arises from such alternative methods and we shall pursue the thoroughly straightforward method advocated in the previous paragraphs. This involves the evaluation of integrals (like K_{O+}, J_{F-}, etc.) and it is therefore time to abandon generalities and deal with some specific examples.

However, before going on to consider these we should remark that, having found solutions for c_O, c_F, c_i and h, the absolute temperature T follows at once from the caloric equation of state, namely

$$h = \sum_{\alpha} c_\alpha(h_\alpha + q_\alpha). \tag{32}$$

The sum is taken over all species (i.e. with $\alpha = O, F, i$ and P, the latter indicating the product species). h_α is the thermal part of the specific enthalpy of species α,

$$h_\alpha = \int_0^T C_{p\alpha}\, dT \tag{33}$$

($C_{p\alpha}$ is the relevant specific heat at constant pressure) and q_α is the heat of formation.

The expression given in (32) can be written in terms of c_O, c_F and c_i because

$$c_P = 1-(c_O+c_F+c_i). \tag{34}$$

Although it is not generally true that $C_{p\alpha}$ is either the same for all α or invariant with T, we shall, for purposes of illustration, assume that the specific heat has both of these features. It follows that, in such a case,

$$h = C_p T + (1-c_O-c_F-c_i)q_P. \tag{35}$$

(We have put $q_O = q_F = q_i = 0$; the heat of formation of the product species, q_P, is negative.) In particular, the flame sheet temperature, T_f, is given by the expression

$$C_p T_f = h_f - (1-c_{if})q_P, \tag{36}$$

since c_O and c_F are both zero on the sheet. The subscript f on h and c_i indicates the value of these quantities where $y = y_f$.

4. Examples

(*i*) *Free-stream problems* $(0 < x < t)$

Assume that the free-stream distributions of oxidant and fuel mass fractions are as follows:

$$\begin{aligned} c_O(0,y,t) &= \chi(t); \quad (0 < y < y_+); \\ &= 0; \quad (y_+ < y). \end{aligned} \tag{37}$$

$$\begin{aligned} c_F(0,y,t) &= \Gamma(t); \quad (0 > y > -y_-); \\ &= 0; \quad (-y_- > y). \end{aligned} \tag{38}$$

Using (31) in the first form of (24), together with the definitions (26), we have

$$2\pi^{\frac{1}{2}} c_O(x,y,t) = \frac{1}{4\pi^{\frac{1}{2}}}\int_0^x \{K_{O+}^{(0)} - \mathscr{M}K_{F+}^{(0)}\}\, y\, \frac{\exp\{-y^2/4(x-x_0)\}}{(x-x_0)^{\frac{3}{2}}}\, dx_0 + K_{O-}, \tag{39}$$

where $K_{O+}^{(0)}$ and $K_{F+}^{(0)}$ in the integrand are found from (26) with x_0 written for x and x_0+t-x written for t.

With the conditions given in (37) and (38),

$$K_{O+}^{(0)} = 2\pi^{\frac{1}{2}}\chi(t-x)\,\mathrm{erf}\,(y_+/2x^{\frac{1}{2}}), \tag{40}$$

$$K_{F+}^{(0)} = 2\pi^{\frac{1}{2}}\Gamma(t-x)\,\mathrm{erf}\,(y_-/2x^{\frac{1}{2}}), \tag{41}$$

$$K_{O-} = \pi^{\frac{1}{2}}\chi(t-x)\{2\,\mathrm{erf}\,(y/2x^{\frac{1}{2}}) + \mathrm{erf}\,[(y_+-y)/2x^{\frac{1}{2}}] - \mathrm{erf}\,[(y_++y)/2x^{\frac{1}{2}}]\}. \tag{42}$$

The results given in (40) and (41) can be substituted into (39) and the integral evaluated (use of the *Faltung* theorem and Laplace transforms is helpful) to give, after some re-arrangements,

$$\begin{aligned} c_O(x,y,t) = {} & \tfrac{1}{2}\chi(t-x)\{\mathrm{erf}\,(y/2x^{\frac{1}{2}}) + \mathrm{erf}\,[(y_+-y)/2x^{\frac{1}{2}}]\} \\ & + \tfrac{1}{2}\mathscr{M}\Gamma(t-x)\{\mathrm{erf}\,(y/2x^{\frac{1}{2}}) - \mathrm{erf}\,[(y_-+y)/2x^{\frac{1}{2}}]\}. \end{aligned} \tag{43}$$

F_O is readily found from (43), so that, making use of (31) and the results already given we discover that the flame location is given by the expression

$$y_f(x,t) = -(\pi x)^{\frac{1}{2}}\left\{\frac{\chi(t-x)\,\mathrm{erf}\,(y_+/2x^{\frac{1}{2}}) - \mathscr{M}\Gamma(t-x)\,\mathrm{erf}\,(y_-/2x^{\frac{1}{2}})}{\chi(t-x)\,[1-\exp\{-y_+^2/4x\}] + \mathscr{M}\Gamma(t-x)[1-\exp\{-y_-^2/4x\}]}\right\}. \tag{44}$$

The solution for y_f takes on a slightly more tractable form if

$$y_+ = y_-. \tag{45}$$

Then we can define
$$W(t-x) \equiv \frac{\chi(t-x) - \mathscr{M}\Gamma(t-x)}{\chi(t-x) + \mathscr{M}\Gamma(t-x)}, \tag{46}$$

$$S(x) \equiv \frac{(\pi x)^{\frac{1}{2}}\,\mathrm{erf}\,(y_+/2x^{\frac{1}{2}})}{1-\exp\{-y_+^2/4x\}}, \tag{47}$$

and write
$$y_f(x,t) = -W(t-x)\,S(x). \tag{48}$$

In this form the flame location is revealed as a steady shape-function $S(x)$, modulated by the free-stream conditions evaluated at an appropriately retarded time. Some particular results are noteworthy. If y_+ is infinitely large, $S(x)$ simplifies to $(\pi x)^{\frac{1}{2}}$ and

$$y_f = -W(t-x)(\pi x)^{\frac{1}{2}}. \tag{49}$$

If, in addition, W does not depend on $t-x$ the flame shape is parabolic and steady. This agrees with (3) where we can now identify a in that equation with

$$-(\pi\kappa/U)^{\frac{1}{2}}W.$$

It is interesting to observe that the field need not be steady for W to be independent of $t-x$. Equation (40) shows that it is only necessary for Γ and χ to be proportional to one another; the free stream fuel and oxidant concentrations must therefore vary similarly with time. The effect will be to produce a flame of steady shape but, as we shall see, varying temperature. These results will be true for any value of y_+ (not just infinity) as can be seen from (46), (47) and (48).

If y_+ is finite, so that the fuel and oxidant streams issue from parallel slots of equal cross-section (as is often found in experimental equipment) then there will always be an x large enough to make $(y_+/2x^{\frac{1}{2}}) \ll 1$. In that event

$$S(x) \sim 4xy_+^{-1}, \tag{50}$$

and the flame shape departs considerably from the parabolic form discussed above.

Before drawing any further inferences from the result given in (50) we should inquire into the effect of the assumption which leads to conditions (10) and (11). This involves the transfer of boundary values from $y = y_f$ to $y = 0$ and requires that the remainders in Taylor series expansions of oxidant and fuel fractions should be negligible; to be explicit we should find

$$\tfrac{1}{2}y_f \frac{\partial^2}{\partial y^2} c_O(x, \theta y_f, t) \ll \frac{\partial}{\partial y} c_O(x, 0+, t),$$

where $0 \leqslant \theta \leqslant 1$. When y_+ is infinite this condition can always be met by ensuring that $W^2 \ll 1$. However, when y_+ is finite this condition will only prove satisfactory if (roughly speaking) $|y_f|$ does not approach $|y_+|$ too closely; this will be satisfied by flames with an asymptotic steady shape like $S(x)$ in (50) if $|W| \ll y_+^2/4x$ and we infer that the present solutions must begin to lose accuracy for sufficiently large x.

One case where this problem will not arise for finite y_+ occurs when W is zero; (40) shows that, in such circumstances,

$$\chi(t) = \mathscr{M}\Gamma(t)$$

and the streams must remain in stoichiometric proportion for all time.

To summarize the results derived so far, we can say that the flame sheet appears to respond instantaneously to the changes of composition in the free stream, in the following way. Any changes which occur in the free streams are signalled to downstream points at the constant convection speed U. Then, at any given point, the flame sheet is found in the position that it should occupy in a steady flow whose free stream composition corresponds with the instantaneous 'local' value. In this sense the sheet can be considered stable in response to perturbations of free stream composition.

To conclude this section, we shall examine the temperature field on the assumption that (35) is an adequate caloric equation of state. It is therefore necessary to calculate c_i, for which purpose we note that, since there is to be no product species in either free stream, conditions (37) and (38) require

$$\left.\begin{aligned} c_i &= 1; \quad y_+ < y; \\ &= 1-\chi(t); \quad 0 < y < y_+; \\ &= 1-\Gamma(t); \quad -y_- < y < 0; \\ &= 1; \quad y < -y_-. \end{aligned}\right\} \tag{51}$$

Putting these results in (21) (with $x < t$) gives

$$\begin{aligned} 1-c_i(x,y,t) = {} & \tfrac{1}{2}\Gamma(t-x)\{\operatorname{erf}[(y_-+y)/2x^{\frac{1}{2}}]-\operatorname{erf}(y/2x^{\frac{1}{2}})\} \\ & +\tfrac{1}{2}\chi(t-x)\{\operatorname{erf}[(y_+-y)/2x^{\frac{1}{2}}]+\operatorname{erf}(y/2x^{\frac{1}{2}})\}. \end{aligned} \tag{52}$$

Since c_P is zero in the free stream, (34) and (35) show that h is equal to $C_p T$ there; we assume that both streams are at the same uniform temperature T_∞, whence we can write, for all y,

$$h(0,y,t) = C_p T_\infty. \tag{53}$$

Equation (21) then shows that

$$h(x,y,t) = C_p T_\infty, \tag{54}$$

for all x in $0 < x < t$; in particular h_f has this value. Equation (36) shows that the increase of T_f above T_∞ is directly proportional to $1-c_i$ in (52) with $y = y_f$. In particular we note how dilution due to diffusion of the inert species will generally cause the flame temperature T_f to fall as x increases along the flame sheet; also T_f only depends on time in its retarded form $t-x$.

(*ii*) *Initial-value problems* ($0 < t < x$)

We shall consider a representative initial distribution for c_O and c_F which, by a choice of the constants involved, can be made to display several features of the initial value problem. The general solutions have already been given in §3 and we observe that it is only the 'J' integrals that appear when $x > t$.

One would anticipate that the use of values for $c_\alpha(x-t, y_0, 0)$ in (25) which correspond to an initial steady state field should, eventually, lead to a solution $c_\alpha(x,y,t)$ entirely similar to the initial value data, since there is no perturbation mechanism present in such circumstances. Confirmation of this fact would provide an interesting check on the analysis, so that we select the following initial distributions for c_O and c_F:

$$c_O(x,y,0) = a+b\operatorname{erf}(y/2x^{\frac{1}{2}}), \tag{55}$$

$$c_F(x,y,0) = d+g\operatorname{erf}(y/2x^{\frac{1}{2}}). \tag{56}$$

(a here need not be confused with the symbol used earlier, since the latter will not arise in this section.) By manipulating the constants a, b, d and g we can include the (special) steady field case amongst a number of other interesting situations.

When using the initial conditions given by (55) and (56) to evaluate $J_{\alpha\mp}$ one must write y_0 for y and $x-t$ for x. The constant terms a and d in (55) and (56) are easily dealt with; the error function parts of these equations are rather more awkward and evaluation of the 'J' integrals is facilitated by judicious differentiations with respect to parameters in their integrands. By these means it can be shown that

$$J_{O-} = 2\pi^{\frac{1}{2}}\{a \operatorname{erf}(y/2t^{\frac{1}{2}}) + b \operatorname{erf}(y/2x^{\frac{1}{2}})\}, \tag{57}$$

$$J_{F-} = -2\pi^{\frac{1}{2}}\{d \operatorname{erf}(y/2t^{\frac{1}{2}}) - g \operatorname{erf}(y/2x^{\frac{1}{2}})\}, \tag{58}$$

$$J^{(0)}_{O+} = 2\pi^{\frac{1}{2}}a + \pi^{\frac{1}{2}}b + \frac{2b}{\pi^{\frac{1}{2}}}\sin^{-1}\left(\frac{2t}{x} - 1\right), \tag{59}$$

$$J^{(0)}_{F+} = 2\pi^{\frac{1}{2}}d - \pi^{\frac{1}{2}}g - \frac{2g}{\pi^{\frac{1}{2}}}\sin^{-1}\left(\frac{2t}{x} - 1\right). \tag{60}$$

From (31) we find that

$$c_O(x, 0+, t) = \tfrac{1}{2}(a - \mathscr{M}d) + \tfrac{1}{4}(b + \mathscr{M}g)\left\{1 + \frac{2}{\pi}\sin^{-1}\left(\frac{2t}{x} - 1\right)\right\}. \tag{61}$$

Writing x_0 for x and $x_0 + t - x$ for t in (61) gives the boundary value required for the first integral in (24), whose limits are now $x - t$ to x. After some rearrangements it is found that

$$\begin{aligned} c_O(x, y, t) = {} & a \operatorname{erf}(y/2t^{\frac{1}{2}}) + \tfrac{1}{2}(a - \mathscr{M}d) \operatorname{erfc}(y/2t^{\frac{1}{2}}) + b \operatorname{erf}(y/2x^{\frac{1}{2}}) \\ & + \frac{1}{2\pi}(b + \mathscr{M}g)\int_0^t \frac{\exp\{-y^2/4u\}}{u^{\frac{1}{2}}(x-u)^{\frac{1}{2}}}\,du. \end{aligned} \tag{62}$$

We can now check whether c_O in (62) will reduce to c_O in (55) if the constants a, b, d and g represent an initial steady state. The latter is ensured if (8) and (9) are satisfied or, in other words, if

$$a = -b \operatorname{erf}(y_f/2x^{\frac{1}{2}}); \quad d = -g \operatorname{erf}(y_f/2x^{\frac{1}{2}}); \tag{63a}$$

$$b = -\mathscr{M}g. \tag{63b}$$

Apart from the fact that the initial value of y_f must be proportional to $x^{\frac{1}{2}}$ (see equation (49) *et seq.*) (63*a*, *b*) also show that

$$a + \mathscr{M}d = 0. \tag{63c}$$

It is clear that (62) does indeed reduce to (55) for all t.

In order to find y_f in conditions which are more general than those of the previous paragraph we now find F_0 from (62):

$$F_0 = \frac{1}{2(\pi t)^{\frac{1}{2}}}(a + \mathscr{M}d) + \frac{1}{2(\pi x)^{\frac{1}{2}}}(b - \mathscr{M}g). \tag{64}$$

From (31) and (61) it follows that

$$y_f = -(\pi x t)^{\frac{1}{2}}\left\{\frac{(a - \mathscr{M}d) + \frac{1}{2}(b + \mathscr{M}g)\,[1 + (2/\pi)\sin^{-1}([2t/x] - 1)]}{(a + \mathscr{M}d)x^{\frac{1}{2}} + (b - \mathscr{M}g)t^{\frac{1}{2}}}\right\}. \tag{65}$$

Let us consider some special cases. First, assume that, prior to the initial instant of time, both fuel and oxidant are uniformly distributed in their respective half-planes ($y < 0$ and $y > 0$). Then both b and g are zero and (65) becomes

$$y_f = -\left\{\frac{a-\mathscr{M}d}{a+\mathscr{M}d}\right\}(\pi t)^{\frac{1}{2}} \equiv y_{fu}. \tag{66}$$

The flame sheet therefore moves bodily (in the region $x > t$) downwards or upwards, depending on whether $a \gtrless \mathscr{M}d$,[†] at a rate which diminishes like $t^{-\frac{1}{2}}$. From the definitions of the dimensionless co-ordinates (see (5)) it is clear that the real displacement is independent of U and that the same result should hold if U is zero; this can readily be confirmed. The (rather idealized) situation requires that, at the same time as the partitition (which can be presumed to exist between the fuel and oxidant prior to $t = 0$) is removed, the whole of the interface ignites. This could possibly be achieved with the use of a thin metal diaphragm separating the gases which is suddenly heated and vapourised.

If t is allowed to approach zero for a fixed x we observe that

$$\sin^{-1}([2t/x]-1) \to -\pi/2,$$

and the term $b+\mathscr{M}g$ in (65) vanishes. It follows that

$$y_f \to y_{fu}; \quad t \downarrow 0. \tag{67}$$

(y_{fu} is defined in (66)). For initial fuel and oxidant mass fractions given by (55) and (56) it follows that the flame sheet always starts from the $y = 0$ axis (unless conditions (63) hold) and that, in the early stages, it begins to move like the flame sheet discussed in the previous paragraph.

Suppose now that the free-stream conditions are consistent with (55) and (56). By this we mean that

$$c_O(0,y,t) = a+b; \quad (y > 0, t \geqslant 0), \tag{68a}$$

$$c_F(0,y,t) = d-g; \quad (y < 0, t \geqslant 0). \tag{68b}$$

In the notation of § 4(i) this means that y_+ and y_- are infinite (so that (49) holds) and that we should interpret χ as $a+b$ and Γ as $d-g$. The flame shape in $0 < x < t$ is therefore given by

$$y_f = -(\pi x)^{\frac{1}{2}}\left\{\frac{(a+b)-\mathscr{M}(d-g)}{(a+b)+\mathscr{M}(d-g)}\right\}. \tag{69}$$

If we now let $x \uparrow t$ in (69) and $x \downarrow t$ in (65) it is easily seen that y_f is continuous across the wave-head $x = t$ and it is depicted as such in figure 1. It is however, equally clear that the slope of the flame sheet $\partial y_f/\partial x$ is not continuous, in general, at the same location. It is fair to conclude that the terms in c_{Oxx}, etc., which have been neglected, will have a significant part to play in fixing details of the flame shape near $x = t$.

[†] Equality of a and $\mathscr{M}d$ means that oxidant and fuel exist initially in stoichiometric proportions; if $a > \mathscr{M}d$ the mixture is fuel-weak.

We have seen how, in the case of initially uniform distributions of oxidant and fuel, the flame sheet moves with diminishing speed to infinity in one direction or another (see (66)). Suppose that we fix a value of the ratio t/x, say

$$\frac{t}{x} = w \quad (<1). \tag{70}$$

Then (65) gives

$$y_f = -(\pi t)^{\frac{1}{2}} \left\{ \frac{(a - \mathscr{M} d) + \frac{1}{2}(b + \mathscr{M} g)\,[1 + (2/\pi)\sin^{-1}(2w - 1)]}{(a + \mathscr{M} d) + (b - \mathscr{M} g)\, w^{\frac{1}{2}}} \right\}, \tag{71}$$

and we see that y_f behaves, locally, like the flame described by (66); its displacement depends on w and the flame is generally therefore not flat.

We can envisage the solutions in this section as crude models for the way in which a flame sheet responds to finding itself in an environment not initially suited to its *steady* continuation. If we fix attention on some chosen x-location, the sheet begins to move from its position on $y = 0$ towards the position which it would occupy in a steady field (see result (69) *et seq.*). When the wave head arrives, at a time $t = x$, the conditions at x become subject entirely to the free-stream conditions which have been imposed after the initial instant; from time $t = x$ onwards the flame responds instantaneously to free-stream changes in the way that has been described in §4(i). Therefore, in the sense that at some given x the flame sheet's movements remain bounded, one may describe it as responding in a stable way to the perturbations imposed upon it.

The flame temperature in $x > t$ will depend not only on the distributions of oxidant and fuel mass fraction, c_O and c_F, but, as can be seen from (35), (36) and (21), on the initial distributions of the diluent concentration c_i and temperature T as well. The general solutions are available in the equations quoted, but we note one very simple result relating to the case arising when the constants b and g (in (55), (56)) are zero. If we also assume that the constants a and d take the values $a = d = 1$, so that the initial diluent concentration vanishes everywhere, (21) shows that the subsequent diluent concentration is always zero in $x > t$. If it is assumed that the initial enthalpy distribution is equal to $C_p T_\infty$, equation (21) shows that the enthalpy always has this value in $x > t$. It follows at once from (36) that the flame sheet temperature is simply equal to T_∞ plus $(-q_P/C_p)$, i.e. it is the adiabatic flame temperature. This is maintained for all time and does not depend on where the flame is or how quickly it is moving.

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